A sceptical view on decomposition attacks

Claus Diem

University of Leipzig

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Consider the **discrete logarithm problem**:

Given $(G, +)$ and A, B with $B = eA$ for some $e \in \mathbb{N}$ compute such an e!

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5. Output $e := -\frac{\sum_i \gamma_i \beta_i}{\sum_i \gamma_i \alpha_i} \in \mathbb{Z}/N\mathbb{Z}$.

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$$P_1, \ldots, P_m \in \mathcal{F}$$

In elliptic curves

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Recall: $x(P)$ determines $\pm P$.

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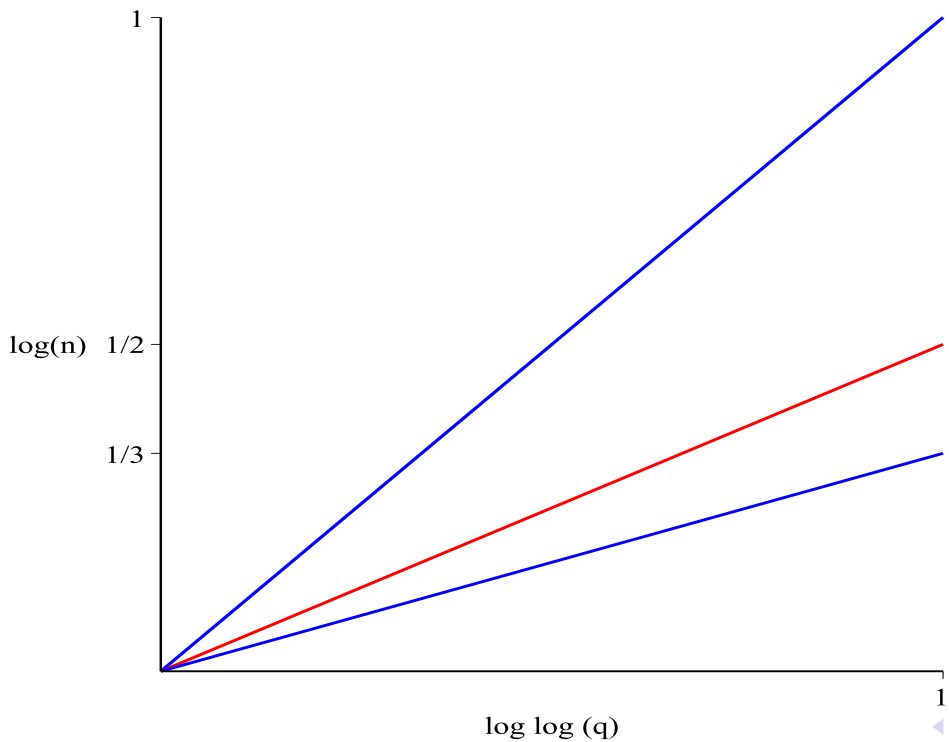
via solving **polynomial systems** over $\mathbb{F}_q$.

A result

Theorem Let $a, b > 0$ be fixed. Then the DLP in elliptic curves over $\mathbb{F}_{q^n}$ can be solved in:

- For $a \cdot \log(q)^{1/2} \leq n \leq b \cdot \log(q)^{1/2}$: $\exp(O(\log(q^n)^{2/3}))$.
- For $a \cdot \log(q)^{1/3} \leq n \leq b \cdot \log(q)$: $\exp(O(\log(q^n)^{3/4}))$.



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1. Linear algebra on a system $X_i^d \cdot F_j \quad$ (variable i, j).
2. Factorization of a single univariate polynomial.

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The running time is usually smaller, but ...

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"Most of the time" it is like considering the system of leading terms.

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- irrelevant if degree q is not reached ("**large characteristic**")
- particularly relevant for $q = 2$.

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Compare: Over $\mathbb{R}$ one can efficiently approximate a single solution.

Large characteristic

Let $E/\mathbb{F}_{q^n}$ be given.

Consider $m = n$, $U = \langle v \rangle$.

We search for decompositions

$$R = P_1 + \cdots + P_n \ , \ x(P_i) \in U \ .$$

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Faster?

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I am going to argue:

If this is correct, then there is a **polynomial time** algorithm.

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Traditional:

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Now set $m := n$, $V_i = \langle v_i \rangle$.

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For points A, B,
find **one** decomposition / representation of the form

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$$S_3(x(B), x(A), x_1) = 0, \;\; S_3(x_1, x(2A), x_2) = 0, \ldots,$$
$$S_3(x_{n-4}, x(2^{n-2}A), x(2^{n-3}A)) = 0$$

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Largest experiment for $n = 11$.

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For $n = 100$: 100^{12} bits $= 10^{24}$ bits $> 10^{21}$ bytes $= 1000^7$ bytes $=$ 1 Zettabyte !