Quantum cryptography with pseudorandom bases and the problem of quantum state discrimination

P.A. Tregubov and A.S. Trushechkin

National Research Nuclear University “MEPhI”
Steklov Mathematical Institute
Russian Quantum Center

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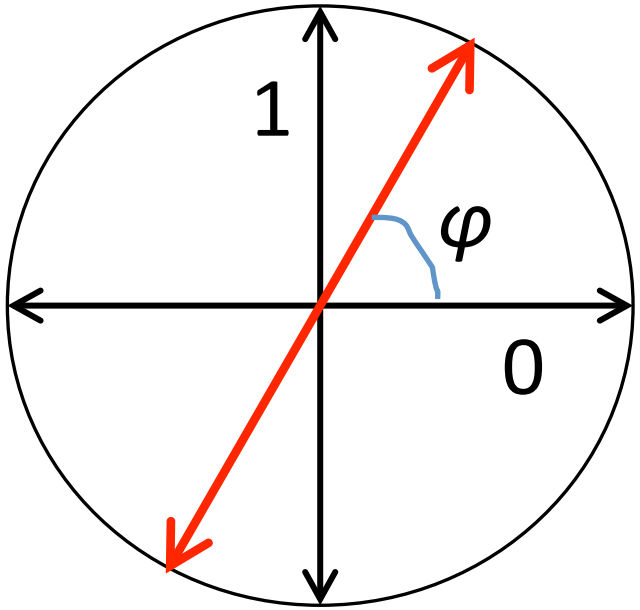
Quantum cryptography

- Quantum cryptography and quantum computations are different things! Quantum cryptography does not need quantum computer
- In most cases, quantum cryptography is in fact quantum key distribution (QKD)
- I.e., if there is no eavesdropping and QKD is successful, Alice and Bob use the key to cipher messages using classical ciphers – GOST, AES, etc.
- If Alice and Bob detect eavesdropping, they abort the QKD protocol. The eavesdropper (Eve) does not obtain useful information, since there was no ciphering with insecure key.

Polarization of photon



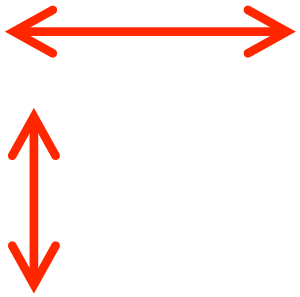
NB:
If you don't know φ,
the measurement
changes the state
(polarization)
of a photon

Probabilities of results:

$p_0 = \cos^2 \varphi$

$p_1 = \sin^2 \varphi$

New polarizations:



The role of quantumness

- If the light (signal) were classical, i.e., ininitely divisible, we could split a tiny part of it and measure this part – we get information with only negligible disturbance of the signal
- A photon is a minimal quantum (portion) of the light, indivisible, and the classical eavesdropping does not work

QKD protocol with pseudorandom bases

- Alice and Bob have an initial (short) secret key k
- They generate a pseudorandom sequence of angles $\varphi_1(k),\ldots, \varphi_N(k)$.
- Alice generates a random bit sequence $x_1,\ldots,x_N$
- Alice sends to Bob photons with polarizations $\varphi_1(k) + x_1\pi/2,\ldots, \varphi_N(k) + x_N\pi/2$
- Bob measures, whether the polarization is $\varphi_i(k)$ or $\varphi_i(k) + \pi/2$, obtains x_i
- Eve does not know $\varphi_i(k)$, hence, if she tries to measure photons, she introduces errors (some Alice's and Bob's values of x_i will differ)

QKD protocol with pseudorandom bases

- Alice and Bob randomly choose a test set T (subset of $\{1,\ldots, N\}$), public their values of x_i for i in T, and count the number of errors
- If the number of errors is too large, then Alice and Bob conclude that eavesdropping took place and abort the protocol
- If not, bits x_i not from T are used to establish a common secret key (error correction and privacy amplification)

Problem

- Rigorously prove the security of the QKD protocol (done for the BB84 protocol)
- In the most general case we assume that Eve obeys unlimited computing powet and quantum thechnologies (quantum memory, quantum computer).
  - She is limited only by the laws of nature
- *Unconditional (information-theoretic) security:* laws of nature + rigorous mathematical proof
- Device-independent proofs of security of QKD: Eve can even control the Alice's and Bob's QKD devices

Multiphoton pulses

- Problem: we cannot guarantee that every signal is strictly single-photon
- If there is more than one photon in the pulse, Eve can take one to her quantum memory
- Problem: how many photons Eve shoud have to calculate the initial key (seed) *k* and, hence, break the system?
- Analogous classical problem: how many terms of pseudorandom sequence are needed to calculate the secret seed?
- NB: we assume infinite computational power of Eve!

Mathematical formulation

- Let l be the length of the initial key (seed)
- Eve need to discriminate between 2^l quantum states, i.e., vectors of dimension 2^n:

$$\psi(k) = \begin{pmatrix} \cos\varphi_1(k) \\ \sin\varphi_1(k) \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} \cos\varphi_n(k) \\ \sin\varphi_n(k) \end{pmatrix}, \quad k \in \{0,1\}^l$$

  i.e., to find optimal measurement of them maximizing the guessing probability.

- Measurement: 2^l of positive-semidefinite $2^n \times 2^n$-matrices $\{M_k\}$:

$$\sum_{k\in\{0,1\}^l} M_k = Id, \quad p_{k'}(k) = \psi(k)^T M_{k'} \psi(k)$$

$$\frac{1}{2^l} \sum_{k\in\{0,1\}^l} p_k(k) = p_{succ}, \quad 1 - p_{succ} = p_{err}$$

Relation to classical mathematical statistics

- Discrimination of quantum states is a quantum analogue of discrimination between different probability distributions

Results

- $p_{err} \geq 1 - 2^{n-l}$ (rigorous result), hence, for $n \ll l$ Eve has practically no chance
- Under heuristic assumption that the pseudorandom sequence is "good" (indistinguishable from random):
- $p_{succ} \geq 1/(1 + 2^{l-n})$
- $p_{succ} \geq 1/2$ for n $\geq l$, hence, Eve breaks the system with non-negligible probability
- Relation to the Marchenko–Pastur law from probability theory

Results

- p_{succ} – probability of the successful guessing of the secret seed by Eve; $p_{err} = 1 - p_{succ}$
- $p_{err} \geq 1 - 2^{n-l}$ (rigorous result), hence, for $n \ll l$ Eve has practically no chance
- Under heuristic assumption that the pseudorandom sequence is "good" (indistinguishable from random):

  $p_{succ} \geq 1/(1 + 2^{l-n})$
- If $n = l - \Delta$, $\Delta \geq 0$, $n \to \infty$, then

  $p_{succ} \geq 2^{\Delta}\{1-2^{\Delta}[1 - 64/(9\pi^2)]\} \approx 2^{\Delta}(1 - 0{,}28\times 2^{\Delta})$
- $p_{succ} \geq 0{,}72$ for $n \geq l$, hence, Eve breaks the system with non-negligible probability

Conclusions

- New quantum key distribution protocol: use of the pseudorandom sequences
- Analysis of its security: quantum analogues of classical problems of breaking the cryptographic PRNGs, discrimination between probability distributions
- Hypothesis: the key generation rate of the new protocol will be greater than that of the BB84 protocol because of additional uncertainty for Eve