Small Scalar Multiplication on Weierstrass Curves using Division Polynomials

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Weierstrass curves and scalar multiplication

Weierstrass curves

(short) Weierstrass curves:

$$E\colon y^2 = x^3 + ax + b, \quad a, b \in \mathbb{F}, \quad 4a^3 + 27b^2 \neq 0,$$

where $\mathbb{F}$ is a large prime finite field.

$\mathbb{F}$-rational points:

$$E(\mathbb{F}) = \{\text{affine points } (x, y) \in \mathbb{F}^2\} \cup \{\text{a special point } O\}.$$

Point addition: $+\colon E(\mathbb{F}) \times E(\mathbb{F}) \to E(\mathbb{F})$:

- defined by an arithmetic circuit over $\mathbb{F}$;
- a commutative group operation (O is zero);
- negation is not expensive.

Usage in cryptography

A group: $G \in E(\mathbb{F}) \setminus \{O\} \mapsto \mathbb{G} = \langle G \rangle$:

- the order $q = |\mathbb{G}|$ is a large prime;
- the cofactor $|E(\mathbb{F})|/q$ is small (for Weierstrass curves, it can achieve 1 that is convenient for various cryptographic protocols);
- $\mathbb{G}^* = \mathbb{G} \setminus \{O\}$.

The key observation (Miller, 1986; Koblitz, 1987). The DL (Discrete Logarithm) problem in $\mathbb{G}$ seems to be hard and, therefore, $\mathbb{G}$ can be used in cryptography.

Scalar multiplication

Fixed base settings: $d \mapsto dG$, $d \in \{1, 2, \ldots, q-1\}$.

[KeyGen, ElGamal.Sign, Schnorr.Sign]

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Variable base settings: $(d, P) \mapsto dP$, $P \in \mathbb{G}^*$.

[Diffie-Hellman, ElGamal.Verify, Schnorr.Verify]

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Arithmetic circuits for scalar multiplication

Circuit$[d]$

Input: coordinates of P, field constants.

Output: coordinates of dP.

Operations: $\mathtt{I}$ (inversion), $\mathtt{M}$ (multiplication), $\mathtt{S}$ (squaring), $\mathtt{A}$ (addition or subtraction), $\mathtt{m}$ (multiplication by a small constant); $\mathtt{h}$ (division by 2).

Requirements:

- the complexity of **Circuit**$[d]$ should be as small as possible (efficiency);
- the structure and complexity of **Circuit**$[d]$ must not depend on d (the constant-time property, security).

Heuristics for the complexity:
$h = \left(\frac{\mathtt{I}}{\mathtt{M}}, \frac{\mathtt{S}}{\mathtt{M}}, \frac{\mathtt{A}\approx\mathtt{m}\approx\mathtt{h}}{\mathtt{M}}\right) \in [80; 100] \times [0.6; 1.0] \times [0; 0.1]$.

$\mathtt{M}$-**complexity,** $\mathtt{M}(h)$: express the complexity as the number of $\mathtt{M}$ operations provided that a heuristic h is used.

Projective coordinates

Jacobian coordinates: $(X, Y, Z) \sim (x, y) = (X/Z^2, Y/Z^3)$.

Homogeneous coordinates: $(X, Y, Z) \sim (x, y) = (X/Z, Y/Z)$.

The coordinate Z acts as a normalizing factor that "absorbs" the expensive operation $\mathtt{I}$.

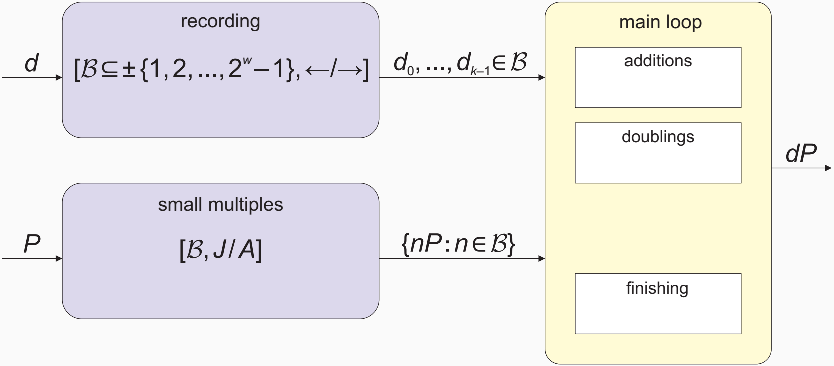
operation ($a = -3$)	multiplicative complexity	notes
$A \leftarrow A + A$	$1\mathtt{I} + 2\mathtt{M} + 1\mathtt{S}$	chord
$A \leftarrow 2A$	$1\mathtt{I} + 2\mathtt{M} + 2\mathtt{S}$	tangent
$J \leftarrow J + J$	$11\mathtt{M} + 5\mathtt{S}$	EFD[add-2007-bl]
$J \leftarrow J + A$	$7\mathtt{M} + 4\mathtt{S}$	EFD[dbl-1998-hnm]
$J \leftarrow 2J$	$3\mathtt{M} + 5\mathtt{S}$	EFD[madd-2007-bl]
$H \leftarrow H + H$	$14\mathtt{M}$	[RCB16], complete
$H \leftarrow H + A$	$13\mathtt{M}$	[RCB16], complete

complete formulas = both addition and doubling

EFD = Elliptic Formula Database (D. Bernstein, T. Lange)

RCB = J. Renes, C. Costello, L. Batina

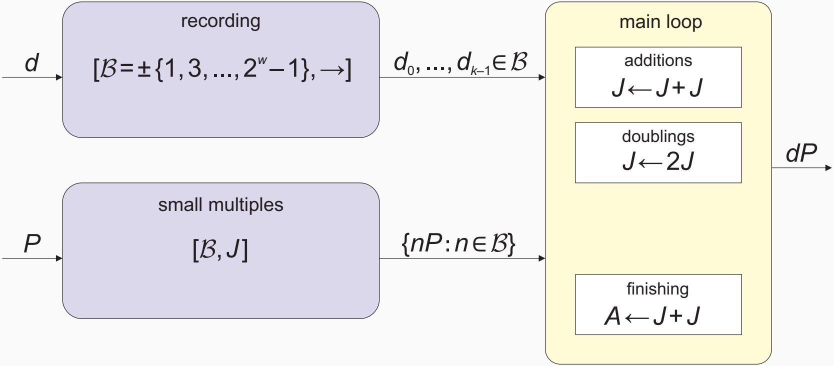
Window methods



Options:

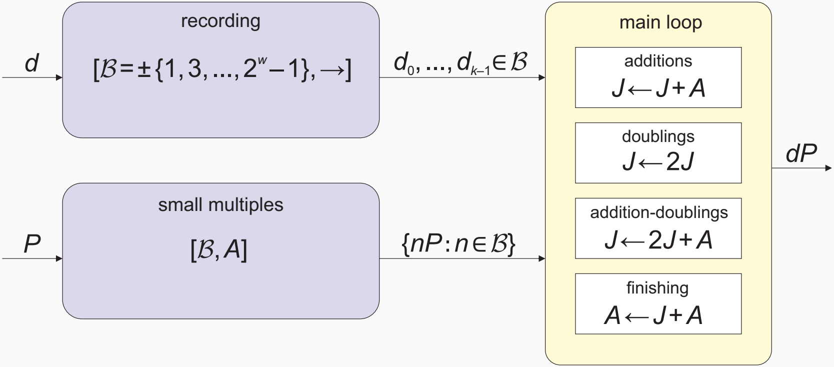
- w (window width);
- $\mathcal{B}$ (base);
- a direction of recording (recoding) the bits of d: $\rightarrow$ or $\leftarrow$;
- coordinates of small multiples: Jacobian or affine.

Strategy I



d
recording
$[\mathcal{B} = \pm\{1, 3, \ldots, 2^w - 1\}, \rightarrow]$
$d_0, \ldots, d_{k-1} \in \mathcal{B}$
P
small multiples
$[\mathcal{B}, J]$
$\{nP : n \in \mathcal{B}\}$
main loop
additions
$J \leftarrow J + J$
doublings
$J \leftarrow 2J$
finishing
$A \leftarrow J + J$
dP

Strategy II


recording
$[\mathcal{B} = \pm\{1, 3, \ldots, 2^w - 1\}, \rightarrow]$
d
$d_0, \ldots, d_{k-1} \in \mathcal{B}$
small multiples
$[\mathcal{B}, A]$
P
$\{nP : n \in \mathcal{B}\}$
main loop
additions
$J \leftarrow J + A$
doublings
$J \leftarrow 2J$
addition-doublings
$J \leftarrow 2J + A$
finishing
$A \leftarrow J + A$
dP

Division polynomials and small scalar multiplication

Division polynomials

If $P = (x, y)$ is an affine point, $n \geq 2$ and $nP \neq O$, then

$$nP = \left(x - \frac{\psi_{n-1}(P)\psi_{n+1}(P)}{\psi_n(P)^2}, \frac{\psi_{n+2}(P)\psi_{n-1}(P)^2 - \psi_{n-2}(P)\psi_{n+1}(P)^2}{4y\psi_n(P)^3} \right).$$

Here $\{\psi_n(P) = \psi_n(x, y)\}$ is **division polynomials**.

Related concepts: elliptic divisibility sequences (Ward 1947), elliptic nets (Stange, 2007).

Usage for scalar multiplication:

- V. Miller (1986): "we may calculate the coordinates of the above points in $26 \log_2 n$ multiplication";
- N. Kanayama et al. (2014) based on Stange (2007): double-and-add, $26\mathtt{M} + 6\mathtt{S}$ per bit (impractical);
- R. Chen et al. (2015): $22\mathtt{M} + 6\mathtt{S}$ per bit (impractical);
- our approach: use division polynomials only for small scalar multiplication.

Recursion

Auxiliary polynomials $W_n(x, y^2)$:

$$\psi_n(x, y) = \begin{cases} 2yW_n(x, y^2), & n \text{ is even}, \\ W_n(x, y^2), & n \text{ is odd}. \end{cases}$$

Assuming (x, y) is fixed, denote $W_n = W_n(x, y^2)$. With that,

$$W_n = \begin{cases} -1, & n = -1, \\ 0, & n = 0, \\ 1, & n = 1, \\ 1, & n = 2, \\ 3(x^2 + a)^2 + 4(3bx - a^2), & n = 3, \\ 2(x^4(x^2 + 5a) + bx(20x^2 - 4a) - 5a^2x^2 - 8b^2 - a^3), & n = 4, \\ W_m \left(W_{m+2} W_{m-1}^2 - W_{m-2} W_{m+1}^2\right), & n = 2m, \\ \left((2y)^4 W_m W_{m+2}\right) W_m^2 - \left(W_{m-1} W_{m+1}\right) W_{m+1}^2, & n = 4k + 1, \quad m = 2k, \\ \left(W_m W_{m+2}\right) W_m^2 - \left((2y)^4 W_{m-1} W_{m+1}\right) W_{m+1}^2, & n = 4k + 3, \quad m = 2k + 1. \end{cases}$$

`SmallMultJ`: Jacobian small multiples (Strategy I)

For $n = 3, 5, \ldots, 2^w - 1$:

$$\begin{aligned} nP &= (X_n, Y_n, W_n), \\ X_n &= xW_n^2 - (2y)^2 W_{n-1} W_{n+1}, \\ Y_n &= y\left(W_{n+2} W_{n-1}^2 - W_{n-2} W_{n+1}^2\right). \end{aligned}$$

Complexity:
$\approx 9.5\mathtt{M} + 3.5\mathtt{S}$ per point.

Naive approach (repeated additions with $2P$):
$\approx 11\mathtt{M} + 5\mathtt{S}$ per point.

SmallMultA: affine small multiples (Strategy II)

For $n = 3, 5, \ldots, 2^w - 1$:

$$nP = \left(\frac{X_n}{W_n^2}, \frac{Y'_n}{W_n^4}\right),$$
$$X_n = xW_n^2 - (2y)^2 W_{n-1} W_{n+1},$$
$$Y'_n = y\left(W_n W_{n+2} W_{n-1}^2 - W_{n-2} W_n W_{n+1}^2\right).$$

Montgomery's trick:

- simultaneous inversion of $\{W_n^2\}$;
- 1I: actually inverse only $\prod_n W_n^2$.

Integration

Recording

1. Write d in base 2^w:

$$d = \sum_{i=0}^{k-1} d_i 2^{wi}, \quad k = \lceil \mathrm{bitlen}(q)/w \rceil, \quad d_i \in \{0, 1, \ldots, 2^w - 1\}. \qquad (\star)$$

2 (d is odd). For $i = k-1, k-2, \ldots, 1$:

$$(d_i, d_{i-1}) \leftarrow (d_i + \mathtt{even}(d_i), d_{i-1} - \mathtt{even}(d_i) 2^w).$$

After that $(\star)$ is preserved but $d_i \in \pm\{1, 3, \ldots, 2^w - 1\}$.

3. If d is even, then

- $d \leftarrow q - d$;
- compute the point $(q - d)P = -dP$;
- negate it.

ScalarMult

Algorithm `ScalarMult`

Input: $P \in \mathbb{G}^*$, $d \in \{1, 2, \ldots, q-1\}$.
Output: dP (in affine coordinates).
Steps:

1. $\delta \leftarrow d \bmod 2$, $d \leftarrow (1-\delta)q + (2\delta - 1)d$.
2. Choose a window width w $(3 \leq w < \log_2 q)$.
3. $P[1] \leftarrow P$, $(P[3], P[5], \ldots, P[2^w - 1]) \leftarrow \texttt{alg}(P, w)$, $\texttt{alg} \in \{\texttt{SmallMultJ}, \texttt{SmallMultA}\}$.
4. $(P[-1], P[-3], \ldots, P[-2^w + 1]) \leftarrow (-P[1], -P[3], \ldots, -P[2^w - 1])$.
5. Represent d as $\sum_{i=0}^{k-1} d_i 2^{wi}$, $d_0, d_1, \ldots, d_{k-1} \in \{0, 1, \ldots, 2^w - 1\}$.
6. $(d_{k-1}, d_{k-2}) \leftarrow (d_{k-1} + \mathrm{even}(d_{k-1}), d_{k-2} - \mathrm{even}(d_{k-1})2^w)$.
7. $Q \leftarrow P[d_{k-1}]$.
8. For $i = k-2, k-3, \ldots, 1$:
    1) $(d_i, d_{i-1}) \leftarrow (d_i + \mathrm{even}(d_i), d_{i-1} - \mathrm{even}(d_i)2^w)$;
    2) $Q \leftarrow 2^w Q$ ($J \leftarrow 2J$, w times);
    3) $Q \leftarrow Q + P[d_i]$ ($J \leftarrow J + J$ or $J \leftarrow J + A$).
9. $Q \leftarrow 2^w Q$.
10. $Q \leftarrow Q + P[d_0]$ ($A \leftarrow J + J$ or $A \leftarrow J + A$).
11. $Q \leftarrow (-1)^\delta Q$.
12. Return Q.

Exceptional cases

Exception #1: addition with O:

- avoided (easy analysis).

Exception #2: doubling instead of addition:

- can be only at Step 10;
- …when $d = q - 2\delta$, $\delta \in \{1, 2, \ldots, 2^w - 1\}$ such that $2^w \mid (q - \delta)$, $2^{w+1} \nmid (q - \delta)$;
- blocked by complete formulas $A \leftarrow J + J/A$ (through the cascade: $H \leftarrow J$, $H \leftarrow H + H/A$, $A \leftarrow H$).

Tuning

Heuristics $h = \left(\frac{\mathtt{I}}{\mathtt{M}}, \frac{\mathtt{I}}{\mathtt{M}}, \frac{\mathtt{A{\approx}m{\approx}h}}{\mathtt{M}}\right)$ **examined**:

- $(100, 0.8, 0)$;
- $(100, 0.67, 0)$;
- $(100, 0.67, 0.05)$.

Bitlen of q: 256, 384, 512.

Analysis (in view of $\mathtt{M}$-complexity):

- Strategy II is better than Strategy I;
- Strategy II can be even faster if we replace $(J \leftarrow 2J, J \leftarrow J + A)$ with $J \leftarrow 2J + A$;
- for Strategy II, the best window length $w^* \in \{4, 5\}$ (8 or 16 small multiples to store).

Comparison

l	Algorithm	M-complexity (rounded to the nearest integer)		
		M(100, 0.8, 0)	M(100, 0.67, 0)	M(100, 0.67, 0.05)
256	ScalarMult[SmallMultJ, 5]	3046	2807	2989
	ScalarMult[SmallMultA, 4]	**2846**	**2636**	**2830**
	MontLadder[WeierCurve]	**2724**	**2590**	**2731**
	MontLadder[MontCurve]	2200	2067	2182
384	ScalarMult[SmallMultJ, 6]	4382	4032	4297
	ScalarMult[SmallMultA, 5]	**4090**	**3774**	**4053**
	MontLadder[WeierCurve]	**4030**	**3829**	**4041**
	MontLadder[MontCurve]	3250	3050	3223
512	ScalarMult[SmallMultJ, 6]	5741	5274	5626
	ScalarMult[SmallMultA, 5]	**5333**	**4912**	**5284**
	MontLadder[WeierCurve]	**5335**	**5068**	**5350**
	MontLadder[MontCurve]	4299	4033	4264

ScalarMult[SmallMultJ, w^*] — Strategy I.

ScalarMult[SmallMultA, w^*] — Strategy II.

MontLadder[MontCurve] — the Montogomery ladder on short Weierstrass curves (M. Hamburg, 2020).

MontLadder[MontCurve] — the Montogomery ladder on Montgomery curves (P. Montgomery, 1987; RFC 7748; the current champion).

Resources

https://github.com/bcrypto/smult