Nonlinearity of bent functions over finite fields

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Notation and definitions

- Let F_q be a finite field of q elements, where $q = p^m$, $p \in P$, $m \in N^*$ and

  F_q^n – an n-dimensional vector space over the field F_q , where $n \in N^*$.
- Denote by P_q^n the set of all mappings of F_q^n into F_q or functions of q-valued logic of n variables, and by A_q^n its subset of affine mappings.
- Let's assign the affine function

$$a(x) = a_0 \oplus a_1 \otimes x_1 \oplus \ldots \oplus a_n \otimes x_n, \quad (1)$$

  where $a_0, a_1, \ldots, a_n \in F_q$, $\oplus$ and $\otimes$ are addition and multiplication operations in F_q, to the vector $\alpha = (a_0, a_1, \ldots, a_n) \in F_q^{n+1}$. Denote by ρ_f^α the Hamming distance between functions $f(x) \in P_q^n$ and $a(x) \in A_q^n$ in the space $F_q^{q^n}$.
- Define the nonlinearity of the function $f(x) \in P_q^n$ by the formula

$$N_f = \min_{\alpha \in F_q^{n+1}} \rho_f^\alpha. \quad (2)$$

- Functions with the maximum possible value of nonlinearity will be called maximally nonlinear and the class of such functions will be denoted by MN_q^n.

History

1960s

Research of **Soviet and American cryptographers** on nonlinearity of Boolean functions

[Glukhov M. M., "On the approximation of discrete functions by linear functions", Mathematical Aspects of Cryptography, 7:4 (2016), 29-50, In Russian].

1976

Rothaus O.S. for even values of n described a class of Boolean functions, which he called bent functions (*let's denote it* B_2^n), for which all Fourier coefficients in the expansions of the corresponding integer functions of the form $(-1)^{f(x)}$ are equal in absolute value. These functions have the maximum possible nonlinearity equal to $2^{n-1}-2^{n/2-1}$, and in the Boolean case, for even n, $B_2^n = MN_2^n$ is true, and for odd n, $B_2^n = \emptyset$ is true. ("1-plateaued" nonlinearity equal to $2^{n-1}-2^{n-1/2}$)

[Rothaus O. S., "On "bent" functions", Journal of Combinatorial Theory, Series A, 20:3 (1976), 300-305].

1985

Kumar, P. V., Scholtz, R. A., Welch, L. R. generalized the concept of a bent function to the case of a residue ring $\mathbb{Z}_k$ ($\mathbb{Z}_k$ *is a field for* $k \in \mathbb{P}$)

[Kumar, P. V., Scholtz, R. A., Welch, L. R., "Generalized bent functions and their properties", Journal of Combinatorial Theory, Series A, 40:1 (1985), 90-107].

History (continued)

1994

Ambrosimov A. S. generalized the concept of a bent function to the case of an arbitrary finite field. He gave the definition of the ***q***-valued bent function (*further this definition is used and the class of **q**-valued bent functions is denoted by* B_q^n), described all the quadratic bent functions and counted their number. For ***q* > 2**, in the case of fields of odd characteristic, for odd values of ***n*** ***q***-valued bent functions also exist.

The **generalized Rothaus criterion** was also proved, which states that a necessary and sufficient condition for a ***q***-valued function to be bent is the balance of any of its nontrivial derivatives (*functions for which all nontrivial derivatives are balanced are also called **perfect nonlinear***)

[Ambrosimov A. S., "Properties of bent functions of q-valued logic over finite fields", Discrete Mathematics and Applications, 4:4 (1994), 341-350].

1997

Coulter, R. S., Matthews, R. W. gave a similar bent function definition and presented their proof of the coincidence of the classes of bent functions and completely nonlinear functions in the case of a finite field

[Coulter, R. S., Matthews, R. W., "Bent polynomials over finite fields", Bulletin of The Australian Mathematical Society, 56 (1997), 429-437]

History (continued)

2002

Solodovnikov V. I. generalized the concept of a bent function to the case of an arbitrary finite abelian group (*The definitions of Ambrosimov and also of Coulter and Matthews of the **q**-valued bent function fall under the general definition of Solodovnikov*)

[Solodovnikov V. I., "Bent functions from a finite abelian group into a finite abelian group", Discrete Mathematics and Applications, 12:2 (2002), 111-126].

2004 - ...

Later, a number of papers have appeared on the topic of ***q***-valued bent functions for ***q* > 2**. Special mention should be made of the work of **Carlet C., Ding C.**, in which generalizations of Maiorana-McFarland's and Dillon's families of bent functions were given to the case of an arbitrary finite field. As it turned out, for $q > 2$ an arbitrary **bent function is not necessarily maximally nonlinear** and **the question of nonlinearity remained open**

[Carlet C., Ding C., "Highly nonlinear mappings", Journal of Complexity, 20:2-3 (2004), 205-244].

Recent results

- In the works of the author

[Ryabov V. G., "On the approximation of restrictions of q-valued logic functions to linear manifolds by affine analogs", Discrete Mathematics, 32:4 (2020), 89-102, In Russian] (hereinafter, "On the approximation"),

[Ryabov V. G., "Maximally nonlinear functions over finite fields", Discrete Mathematics, 33:1 (2021), 47-63, In Russian] (hereinafter, "Maximally nonlinear functions")

- for $\forall\, f(x) \in P_q^n$, the following upper bound for nonlinearity was obtained

$$N_f \leq (q-1)q^{n-1} - q^{n/2-1}; \quad (3)$$

- for $n = 1$ it can be refined:

$$N_f \leq q-2; \quad (4)$$

Recent results (continued)

- for $q > 2$ and even n it was shown that the set of quadratic bent functions splits into two classes of Extended-Affine (**EA-**) equivalent functions. For functions of one class, the equality

$$N_f = (q-1)q^{n-1} - q^{n/2-1} \quad (5)$$

holds, and this class **consists of maximally nonlinear functions**, while for functions of another class the equality

$$N_f = (q-1)(q^{n-1} - q^{n/2-1}) \quad (6)$$

holds, and this class **does not contain of maximally nonlinear functions**;

- for odd p and n was shown that all quadratic q-valued bent functions have the same nonlinearity equal to

$$N_f = (q-1)q^{n-1} - q^{(n-1)/2}, \quad (7)$$

and are maximally nonlinear in the case $n = 1$.

- Thus, the equality (5) is **a criterion of maximum nonlinearity for even n**

Necessary condition for maximum nonlinearity

Theorem 1.

*Let **f**(**x**) ∈ MN$_q^n$, where **q** > **2** and **n** is even. Then in the space $\boldsymbol{F}_q^n$ there is no linear manifold of dimension greater than or equal to **n/2** on which the restriction of the function **f**(**x**) coincides with the restriction of some affine function.*

The proof of the theorem is based on the properties of the restrictions of functions of ***q***-valued logic on the linear manifolds of the vector space of the domain of definition, studied in the author's work "On the approximation".

In the case of Boolean functions, Theorem 1 does not work.

Indeed, for Boolean bent functions from Maiorana-McFarland's and Dillon's families, there are always manifolds of dimension **n/2** on which their restrictions coincide with affine functions.

Results for known families of bent functions

The Mayorana-McFarland's construction for ***q***-valued bent functions of ***n*** variables has the following form

$$f(\boldsymbol{x}) = \langle \boldsymbol{x}', \pi(\boldsymbol{x}'') \rangle \oplus g(\boldsymbol{x}''), \quad \textbf{(8)}$$

where $\boldsymbol{x}' = (x_1, \dots, x_{n/2})$, $\boldsymbol{x}'' = (x_{n/2+1}, \dots, x_n) \in \boldsymbol{F}_q^{n/2}$, $\boldsymbol{\pi}$ is an arbitrary substitution on the set $\boldsymbol{F}_q^{n/2}$, $\langle *, * \rangle$ is the scalar product of vectors in the space $\boldsymbol{F}_q^{n/2}$, and ***g*** is an arbitrary function from $\boldsymbol{P}_q^{n/2}$

Corollary 1. *Let* $\boldsymbol{q} > \boldsymbol{2}$, ***n*** *is even and* $\boldsymbol{f(x)}$ *belongs to the Mayorana-MacFarland's family of* ***q****-valued bent functions of* ***n*** *variables. Then* $\boldsymbol{f(x)} \notin \boldsymbol{MN_q^n}$.

Results for known families of bent functions (continued)

Using the correspondence of the vector space $\boldsymbol{F}_q^{n/2}$ to the field $\boldsymbol{F}_{q^{n/2}}$, the Dillon's construction for q-valued bent functions of *n* variables are defined as follows

$$f(\boldsymbol{x}) = h(\boldsymbol{x}' \otimes (\boldsymbol{x}'')^{q^{n/2-2}}), \quad (9)$$

where $\boldsymbol{x}', \boldsymbol{x}'' \in \boldsymbol{F}_{q^{n/2}}$, $\otimes$ and $(*)^*$ are operations of multiplication and exponentiation in the field $\boldsymbol{F}_{q^{n/2}}$, respectively, and the mapping $h : \boldsymbol{F}_{q^{n/2}} \to \boldsymbol{F}_q$ is balanced function.

Corollary 2. *Let* $q > 2$, n *is even and* $f(\boldsymbol{x})$ *belongs to the Dillon's family of q-valued bent functions of* n *variables. Then* $f(\boldsymbol{x}) \notin MN_q^n$.

New construction of maximally nonlinear bent functions

Theorem 2.

Let $q > 2$, n *is even,* $x' = (x_1, x_2) \in F_q^{n/2}$, $x'' = (x_3, \ldots, x_{n/2+1})$, $x''' = (x_{n/2+2}, \ldots, x_n) \in F_q^{n/2-1}$, π *is an arbitrary substitution on the set* $F_q^{n/2}$, $\langle *, * \rangle$ *is the scalar product of vectors in the space* $F_q^{n/2}$, *and* g *is an arbitrary function from* $P_q^{n/2}$. *Then*

a) for fields of even characteristic with respect to the function

$$f(x) = x_1 \otimes x_2 \oplus x_1^2 \oplus c \otimes x_2^2 \oplus \langle x'', \pi(x''') \rangle \oplus g(x'''), \quad (10)$$

where c *is a free term of an irreducible polynomial* $x^2 \oplus x \oplus c$, *the statements* $f(x) \in B_q^n$ *and* $f(x) \in MN_q^n$ *are true;*

b) for fields of odd characteristic with respect to the function

$$f(x) = x_1^2 \ominus d \otimes x_2^2 \oplus \langle x'', \pi(x''') \rangle \oplus g(x'''), \quad (11)$$

where d *is a quadratic nonresidue, the statements* $f(x) \in B_q^n$ *and* $f(x) \in MN_q^n$ *are true.*

New construction (continued)

- The proof of the theorem is based
  - on the properties of the restrictions of functions of ***q***-valued logic on the linear manifolds, studied in the author's work “On the approximation”,
  - as well as on the results obtained in another author's work “Maximally nonlinear functions” that

    - for fields of even characteristic the quadratic form

    $$x_1 \otimes x_2 \oplus x_1^2 \oplus c \otimes x_2^2, \quad (12)$$

    where c is a free term of an irreducible polynomial $x^2 \oplus x \oplus c$, is a maximally nonlinear bent function, and

    - for fields of odd characteristic the quadratic form

    $$x_1^2 \ominus d \otimes x_2^2, \quad (13)$$

    where d is a quadratic nonresidue, also is a maximally nonlinear bent function.

New construction (continued)

- In similar ways, one can show that for $q > 2$ and n is odd, for the function

$$f(x) = x_1^2 \oplus \langle x'', \pi(x''') \rangle \oplus g(x''') \quad (14)$$

the following inequality holds

$$N_f \geq (q-1)q^{n-1} - q^{(n-1)/2}, \quad (15)$$

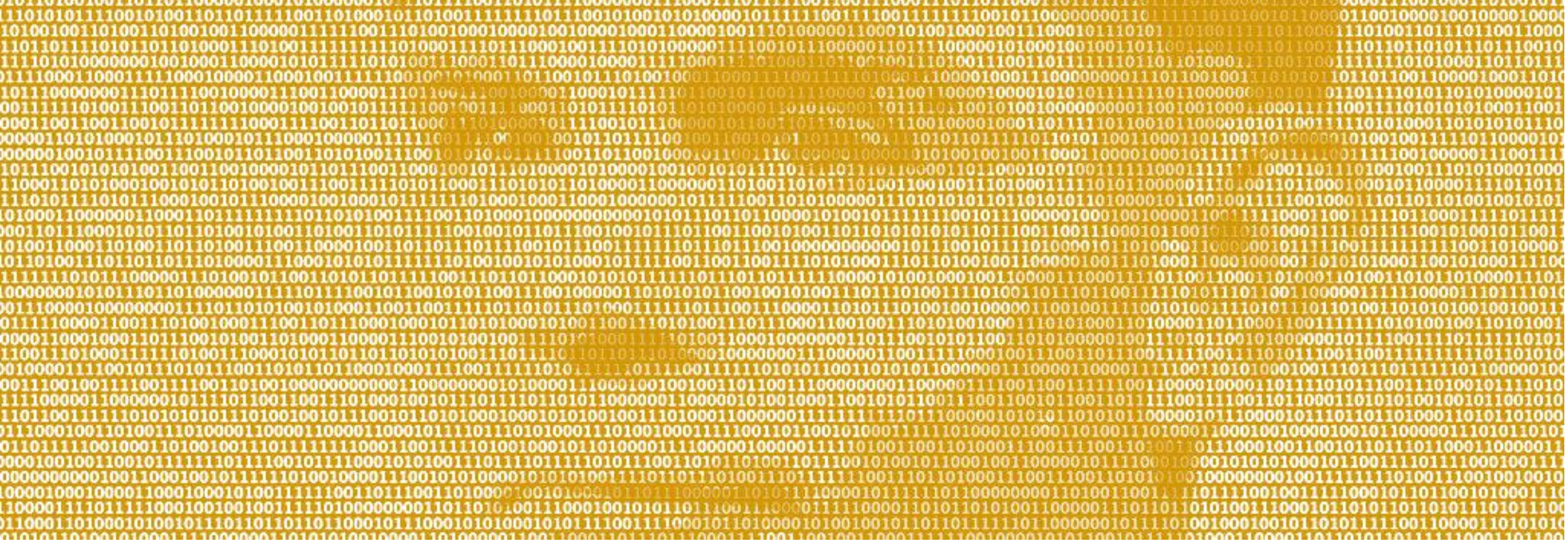
and for $n = 1$ this function is maximally nonlinear.

In the case of a field with an odd characteristic, $f(x)$ is a bent function, while for a field of an even characteristic, it is a balanced function and not a bent function.

- The families of functions of q-valued logic constructed above can be extended by adding **EA**-equivalent functions from P_q^n.

Conclusion

- The results obtained here confirm that for $q > 2$ and even values n, some famous families of q-valued bent functions do not possess the property of maximum nonlinearity. At the same time a new family of bent functions over finite fields is constructed, which are **both bent and maximally nonlinear.** Moreover, the functions of this family can have an arbitrary degree of the polynomial **in the range from 2 to max $\{2; (q-1)(n/2-1)\}$**.
- For odd values of n, a family of q-valued functions with a sufficiently high degree of nonlinearity is indicated. In the case of fields of odd characteristic, this family belongs to the class of **bent functions**, and for fields of even characteristic, it belongs to the class of **balanced functions**.



Thank you for your attention!

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